

On the topology of the hypermultiplet moduli space in type II/CY string vacua

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By analyzing qualitative aspects of NS5-brane instanton corrections, we determine the topology of the hypermultiplet moduli space \mathcal{M}_H in Calabi-Yau compactifications of type II string theories at fixed value of the dilaton and of the Calabi-Yau metric. Specifically, we show that for fivebrane instanton couplings to be well-defined, translations along the intermediate Jacobian must induce non-trivial shifts of the Neveu-Schwarz axion which had thus far been overlooked. As a result, the Neveu-Schwarz axion parametrizes the fiber of a circle bundle, isomorphic to the one in which the fivebrane partition function is valued. In a companion paper [1], we go beyond the present analysis and take steps towards a quantitative description of fivebrane instanton corrections, using a combination of mirror symmetry, S-duality, topological string theory and twistor techniques.

INTRODUCTION

Determining the exact, quantum corrected low energy effective action in type II string theories compactified on a Calabi-Yau threefold \mathcal{X} is a challenging problem, with many applications at stake. Supersymmetry requires that the moduli space of massless scalars is locally (and, presumably, also globally) a product $\mathcal{M}_V \times \mathcal{M}_H$ of the vector multiplet (VM) moduli space \mathcal{M}_V and the hypermultiplet (HM) moduli space \mathcal{M}_H [2]. While the former can be computed exactly using classical mirror symmetry, the latter receives non-perturbative instanton corrections [3], which, in the current formulation of string theory, can only be determined indirectly.

Using a combination of physical arguments (T-duality, mirror symmetry and S-duality) and known mathematical structures (twistor techniques for quaternion-Kähler manifolds and wall-crossing formulae for generalized Donaldson-Thomas invariants), D-instanton corrections to the metric on \mathcal{M}_H were recently expressed in terms of the generalized Donaldson-Thomas invariants of \mathcal{X} [4, 5], in close parallel with the description of the HM moduli space of $\mathcal{N} = 2$ super Yang Mills theories on $\mathbb{R}^3 \times S^1$ given in [6]. The metric on \mathcal{M}_H , however, should also receive instanton corrections from NS5-branes wrapped on \mathcal{X} . Those are expected to restore S-duality invariance [7], lift the ambiguity of the D-instanton asymptotic series [8], and resolve the singularity of the perturbative moduli space metric [9].

Part of the difficulty of including fivebrane instantons lies in the fact that the type IIA fivebrane supports a self-dual 3-form flux H , which is inherently quantum mechanical [10, 11]. In particular, H cannot be simultaneously measured on two three-cycles $\gamma, \gamma' \in H_3(\mathcal{X}, \mathbb{Z})$ with non-zero intersection product $\langle \gamma, \gamma' \rangle$. Equivalently, the D2-brane charge is ill-defined in the presence of a fivebrane. As a result, the partition sum over fluxes/D2-branes is not simply a function of the metric on \mathcal{X} and of the back-

ground three-form field C , but rather a section of a certain circle bundle \mathcal{C}_{NS5} over the intermediate Jacobian of \mathcal{X} (a torus bundle over the complex structure moduli space, with fiber $\mathcal{T} = H^3(\mathcal{X}, \mathbb{R})/H^3(\mathcal{X}, \mathbb{Z})$ parametrized by the C -field). The restriction $\mathcal{C}_{\text{NS5}}|_{\mathcal{T}} \equiv \mathcal{C}_{\Theta}$ to the torus \mathcal{T} further depends on a choice of “generalized spin structure” Θ on \mathcal{X} [10].

On the other hand, a fivebrane wrapped on \mathcal{X} is expected to correct the metric on \mathcal{M}_H via a term schematically of the form

$$\delta ds^2|_{\text{NS5}} \sim e^{-4\pi r - \pi i \sigma} \mathcal{Z}_{\Theta}(C), \quad (1)$$

where r is related to the four-dimensional string coupling via $r = 1/g_4^2$, σ is the Neveu-Schwarz (NS) axion, dual to the Kalb-Ramond two-form in four dimensions, and \mathcal{Z}_{Θ} is the aforementioned fivebrane partition function. The consistency of (1) requires that $e^{i\pi\sigma}$ and \mathcal{Z}_{Θ} should be valued in the same circle bundle \mathcal{C}_{NS5} .

In this note, by enforcing the consistency of the NS5-instanton correction (1), we determine the topology of the hypermultiplet moduli space \mathcal{M}_H at fixed (weak) value of the string coupling and fixed complex structure on \mathcal{X} . This result is a crucial prerequisite for a quantitative analysis of fivebrane instanton corrections to the metric on \mathcal{M} , which will be presented in the companion paper [1]. For definiteness, we focus on the HM moduli space in type IIA string theory compactified on a Calabi-Yau threefold \mathcal{X} . By mirror symmetry and T-duality, the same considerations apply to the HM moduli space in type IIB string theory compactified on the mirror threefold $\hat{\mathcal{X}}$, or to the VM moduli space in type IIA (type IIB, respectively) compactified on $\hat{\mathcal{X}} \times S^1$ ($\mathcal{X} \times S^1$, respectively). In [1], to which the reader is referred for more details and references, we extend the schematic coupling (1) to the case of $k > 1$ fivebranes and promote it to an actual deformation of the metric on \mathcal{M}_H by combining insights from mirror symmetry, S-duality, topological string theory and twistor techniques.

PERTURBATIVE HYPERMULTIPLY METRIC

Recall that in type IIA string theory, the HM moduli space describes the vacuum expectation values of the dilaton r and NS-axion σ , introduced in (1), the complex structure of \mathcal{X} and the periods of the 3-form field C on $H_3(\mathcal{X}, \mathbb{Z})$. To write the metric explicitly, let us choose a symplectic basis $(\mathcal{A}^\Lambda, \mathcal{B}_\Lambda)$, $\Lambda = 0, \dots, h_{2,1}(\mathcal{X})$, of $\Gamma \equiv H_3(\mathcal{X}, \mathbb{Z})$ and complex coordinates z^a , $a = 1, \dots, h_{2,1}(\mathcal{X})$, on the complex structure moduli space $\mathcal{M}_c(\mathcal{X})$. In the one-loop approximation, the metric element on \mathcal{M}_H can be written as follows [9, 12, 13]:

$$ds_{\mathcal{M}}^2 = \frac{r+2c}{r^2(r+c)} dr^2 + \frac{4(r+c)}{r} ds_{\mathcal{SK}}^2 + \frac{ds_{\mathcal{T}}^2}{r} \quad (2)$$

$$+ \frac{2c}{r^2} e^\kappa |X^\Lambda d\tilde{\zeta}_\Lambda - F_\Lambda d\zeta^\Lambda|^2 + \frac{r+c}{16r^2(r+2c)} D\sigma^2.$$

Here, $\Omega \equiv (X^\Lambda, F_\Lambda)$ are the complex periods of the holomorphic three-form $\Omega_{3,0}$ along the symplectic basis $(\mathcal{A}^\Lambda, \mathcal{B}_\Lambda)$, $ds_{\mathcal{SK}}^2$ is the special Kähler metric on the complex structure moduli space $\mathcal{M}_c(\mathcal{X})$, with Kähler potential $\mathcal{K} = -\log[i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)]$, $C \equiv (\zeta^\Lambda, \tilde{\zeta}_\Lambda) \in \mathcal{T}$ are the real periods of the C -field on the symplectic basis of $H_3(\mathcal{X}, \mathbb{Z})$,

$$ds_{\mathcal{T}}^2 = -\frac{1}{2}(d\tilde{\zeta}_\Lambda - \mathcal{N}_{\Lambda\Lambda'} d\zeta^{\Lambda'}) \text{Im} \mathcal{N}^{\Lambda\Sigma} (d\tilde{\zeta}_\Sigma - \tilde{\mathcal{N}}_{\Sigma\Sigma'} d\zeta^{\Sigma'}) \quad (3)$$

is the Kähler metric on the intermediate Jacobian torus $\mathcal{T} = H^3(\mathcal{X}, \mathbb{R})/\Gamma$ (where we identify Γ with $H^3(\mathcal{X}, \mathbb{Z})$, neglecting torsion), $\mathcal{N}_{\Lambda\Sigma}$ is the period matrix in the Weil complex structure, related to the Griffiths period matrix $\tau_{\Lambda\Sigma} = \partial_\Lambda \partial_{\Sigma'} F(X)$ ($F(X)$ being the prepotential) via the usual special geometry relation

$$\mathcal{N}_{\Lambda\Lambda'} = \bar{\tau}_{\Lambda\Lambda'} + 2i \frac{[\text{Im} \tau \cdot X]_\Lambda [\text{Im} \tau \cdot X]_{\Lambda'}}{X^\Sigma \text{Im} \tau_{\Sigma\Sigma'} X^{\Sigma'}}, \quad (4)$$

and, finally, $D\sigma$ is the one form

$$D\sigma = d\sigma + \tilde{\zeta}_\Lambda d\zeta^\Lambda - \zeta^\Lambda d\tilde{\zeta}_\Lambda + 8c\mathcal{A}_K, \quad (5)$$

where $\mathcal{A}_K = \frac{i}{2}(\mathcal{K}_a dz^a - \mathcal{K}_{\bar{a}} d\bar{z}^{\bar{a}})$ is the Kähler connection on $\mathcal{M}_c(\mathcal{X})$. The value of the parameter c was determined by a one-loop computation in [14, 15] to be

$$c = -\chi(\mathcal{X})/(192\pi), \quad (6)$$

where $\chi(\mathcal{X})$ is the Euler number of \mathcal{X} . This value was later seen to be consistent with S-duality on the type IIB side [7]. Alternatively, (6) may be derived by dimensionally reducing the topological coupling $B \wedge I_8$ in the low energy effective action of type IIA string theory in 10 dimensions and dualizing B into σ [1]. The actual value of c will only become important at the end of this note.

For any value of the parameter c , the metric (2) is, as required by supersymmetry [2], quaternion-Kähler (though not complete when $c < 0$, due to a curvature singularity at $r = -2c$), and asymptotes to the c -map

metric [16, 17] in the weak coupling limit $r \rightarrow \infty$. It moreover admits a continuous group of isometries

$$T_{H,\kappa} : (\zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) \mapsto (\zeta^\Lambda + \eta^\Lambda, \tilde{\zeta}_\Lambda + \tilde{\eta}_\Lambda, \sigma + 2\kappa - \tilde{\eta}_\Lambda \zeta^\Lambda + \eta^\Lambda \tilde{\zeta}_\Lambda), \quad (7)$$

with $H = (\eta^\Lambda, \tilde{\eta}_\Lambda) \in \mathbb{R}^{b_3(\mathcal{X})}$, $\kappa \in \mathbb{R}$, satisfying the Heisenberg group law

$$T_{H_2, \kappa_2} T_{H_1, \kappa_1} = T_{H_1+H_2, \kappa_1+\kappa_2+\frac{1}{2}\langle H_1, H_2 \rangle}, \quad (8)$$

where $\langle H_1, H_2 \rangle \equiv \tilde{\eta}_{\Lambda,1} \eta_2^\Lambda - \tilde{\eta}_{\Lambda,2} \eta_1^\Lambda$ is the natural symplectic pairing on $H^3(\mathcal{X}, \mathbb{R})$ (for $c = 0$, there is an additional continuous isometry rescaling (r, C, σ) with weights $(1, 1, 2)$ but it is broken when $c \neq 0$). In addition, the metric (2) admits discrete isometries corresponding to monodromies in $\mathcal{M}_c(\mathcal{X})$, accompanied by an integer symplectic action on the vector C and a shift of σ ,

$$C \mapsto \rho(M) \cdot C, \quad \sigma \mapsto \sigma + \frac{\chi(\mathcal{X})}{24\pi} \text{Im} f_M, \quad (9)$$

where f_M is a local holomorphic function on $\mathcal{M}_c(\mathcal{X})$ determined by the rescaling $\Omega_{3,0} \mapsto e^{f_M} \Omega_{3,0}$ undergone by the holomorphic 3-form under monodromy.

The metric (2) is presumed to be valid to all orders in $1/r$, by standard non-renormalization arguments (see e.g. [18]). It does however receive D- and NS5-instanton corrections at order $e^{-\sqrt{r}}$ and e^{-r} , respectively, which break the continuous isometries (7) to a discrete subgroup corresponding to large gauge transformations. Our main goal in this note is to identify this group.

D-INSTANTON CORRECTIONS

In the type IIA set-up, D-instantons originate from Euclidean D2-branes wrapped on a special Lagrangian (sLag) submanifolds of \mathcal{X} (more generally, from stable objects in the Fukaya category of \mathcal{X}). Qualitatively, they induce corrections to the metric (2) of the following form (see [4] for the precise result)

$$\delta ds^2|_{D2} \sim \Omega(\gamma, z^a) \sigma_D(\gamma) e^{-8\pi\sqrt{r}|Z_\gamma|-2\pi i(q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda)}, \quad (10)$$

where (p^Λ, q_Λ) are integers which label the homology class $\gamma = q_\Lambda \mathcal{A}^\Lambda - p^\Lambda \mathcal{B}_\Lambda \in \Gamma$, $Z_\gamma \equiv e^{\mathcal{K}/2}(q_\Lambda X^\Lambda - p^\Lambda F_\Lambda)$ is the volume (or, in mathematical parlance, the stability data) of any sLag in the homology class γ , $\Omega(\gamma, z^a)$ is Joyce's invariant [19] (a particular instance of generalized Donaldson-Thomas invariants), and $\sigma_D(\gamma)$ is a quadratic refinement of the symplectic pairing on $H_3(\mathcal{X}, \mathbb{Z})$, i.e. a phase assignment $\sigma_D : H_3(\mathcal{X}, \mathbb{Z}) \rightarrow U(1)$ such that

$$\sigma_D(\gamma + \gamma') = (-1)^{\langle \gamma, \gamma' \rangle} \sigma_D(\gamma) \sigma_D(\gamma'). \quad (11)$$

As explained in [6], this phase factor is crucial in ensuring consistency with the Kontsevich-Soibelman wall-crossing formula [20], and hence smoothness of the metric across

lines of marginal stability in $\mathcal{M}_c(\mathcal{X})$ where $\Omega(\gamma, z)$ jumps. Having chosen a Lagrangian decomposition $\Gamma = \Gamma_e \oplus \Gamma_m$ into \mathcal{A} and \mathcal{B} cycles, the solutions of (11) can be parametrized by characteristics $\Theta_D = (\theta_D^\Lambda, \phi_{D,\Lambda}) \in \mathcal{T}$ such that [11]

$$\sigma_D(\gamma) = e^{-i\pi q_\Lambda p^\Lambda + 2\pi i(q_\Lambda \theta_D^\Lambda - p^\Lambda \phi_{D,\Lambda})}. \quad (12)$$

The characteristics Θ_D may be set to zero by redefining $\hat{C} = C - \Theta_D$, at the cost of spoiling the transformation property (9) of C under monodromies, as observed in [6]. The essential quadratic term in (12), however, cannot be disposed of.

Leaving the prefactor in (10) aside for now, we see that the D-instanton corrections break the continuous isometries $T_{H,\kappa}$ to a subgroup where $H \in H^3(\mathcal{X}, \mathbb{Z})$ and κ can still take any value in \mathbb{R} . In particular, keeping the dilaton fixed and quotienting out by translations along the NS-axion, the HM moduli space reduces to the intermediate Jacobian of \mathcal{X} [21–23].

THE FIVEBRANE PARTITION FUNCTION

Let us now (re)turn to the NS5-instanton coupling (1). As first discussed in [10], the partition function \mathcal{Z}_Θ of a self-dual three-form on the fivebrane worldvolume \mathcal{X} is a section of a circle bundle \mathcal{C}_{NS5} over the intermediate Jacobian $\mathcal{J}_c(\mathcal{X})$, whose restriction to the torus \mathcal{T} has first Chern class equal to the Kähler class,

$$c_1(\mathcal{C}_{\text{NS5}})|_{\mathcal{T}} = \omega_{\mathcal{T}} \equiv d\tilde{\zeta}_\Lambda \wedge d\zeta^\Lambda. \quad (13)$$

For a fixed metric on \mathcal{X} , such a circle bundle over \mathcal{T} is determined by the holonomies $\sigma_{\text{NS5}}(H) \in U(1)$ around one-cycles in \mathcal{T} (equivalently, three-cycles $H = (\eta^\Lambda, \tilde{\eta}_\Lambda) \in H^3(\mathcal{X}, \mathbb{Z})$), satisfying the relations

$$\sigma_{\text{NS5}}(H + H') = (-1)^{\langle H, H' \rangle} \sigma_{\text{NS5}}(H) \sigma_{\text{NS5}}(H'), \quad (14)$$

identical in form to the defining relation (11) of a quadratic refinement. As before, we can solve (14) using characteristics $\Theta = (\theta, \phi)$, so that

$$\sigma_{\text{NS5}}(H) = e^{-i\pi \tilde{\eta}_\Lambda \eta^\Lambda + 2\pi i(\tilde{\eta}_\Lambda \theta^\Lambda - \eta^\Lambda \phi_\Lambda)}. \quad (15)$$

We shall later argue that σ_D and σ_{NS5} should be chosen to be identical but for now we keep them distinct.

Having chosen the holonomies $\sigma_{\text{NS5}}(H)$, the bundle $\mathcal{C}_\Theta \equiv \mathcal{C}_{\text{NS5}}|_{\mathcal{T}}$ is now defined by the twisted periodicity property of its sections $\mathcal{Z}_\Theta(C)$ under large gauge transformations [11],

$$\mathcal{Z}_\Theta(C + H) = \sigma_{\text{NS5}}(H) e^{i\pi(\eta^\Lambda \tilde{\zeta}_\Lambda - \tilde{\eta}_\Lambda \zeta^\Lambda)} \mathcal{Z}_\Theta(C), \quad (16)$$

for all $H = (\eta^\Lambda, \tilde{\eta}_\Lambda) \in H^3(\mathcal{X}, \mathbb{Z})$. It is easy to see that such sections can always be written as a generalized theta series

$$\mathcal{Z}_\Theta(C) = \sum_{n^\Lambda \in \Gamma_m + \theta} \Psi(\zeta^\Lambda - n^\Lambda) e^{2\pi i(\tilde{\zeta}_\Lambda - \phi_\Lambda)n^\Lambda + i\pi(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde{\zeta}_\Lambda)}, \quad (17)$$

where the kernel $\Psi(\zeta^\Lambda)$ is an arbitrary function on $\Gamma_m \otimes \mathbb{R}$, which may in general depend on the metric of \mathcal{X} . In particular, starting from the partition function of a non-chiral Gaussian three-form on \mathcal{X} and performing holomorphic factorization [10, 11, 24, 25] leads to a particular solution of (16) given by a Gaussian kernel

$$\Psi(\zeta^\Lambda) = \mathcal{F} \exp(i\pi \zeta^\Lambda \tilde{\mathcal{N}}_{\Lambda\Sigma} \zeta^\Sigma), \quad (18)$$

where \mathcal{F} is a normalization factor which depends on the complex structure of \mathcal{X} . This solution is proportional to the standard Siegel theta series of rank $b_3(\mathcal{X})/2$, and has the additional feature of being holomorphic with respect to the Weil complex structure on \mathcal{T} . The exact fivebrane partition function, which we compute in [1] using S-duality and twistorial techniques, is in general non-Gaussian and non-holomorphic, but it does reduce to this solution in the weak coupling limit. In particular, substituting (18) in (1), we recover the classical fivebrane instanton action expected from the supergravity analysis of [26],

$$S_{\text{NS5}} = \pi [4r - i(n^\Lambda - \zeta^\Lambda) \tilde{\mathcal{N}}_{\Lambda\Sigma} (n^\Sigma - \zeta^\Sigma)] + i\pi [\sigma + \zeta^\Lambda \tilde{\zeta}_\Lambda - 2n^\Lambda (\tilde{\zeta}_\Lambda - \phi_\Lambda) - \theta^\Lambda \phi_\Lambda]. \quad (19)$$

At this stage we wish to stress two important points. First, while the bundles \mathcal{C}_Θ and $\mathcal{C}_{\Theta'}$ are not isomorphic when $\Theta - \Theta' \notin \Gamma$, the corresponding theta series for the same kernel $\Psi(\zeta)$ are nevertheless related by a simple shift of C ,

$$\mathcal{Z}_\Theta(C) = e^{i\pi(\langle \Theta, \Theta' \rangle + \langle C, \Theta - \Theta' \rangle)} \mathcal{Z}_{\Theta'}(C + \Theta' - \Theta). \quad (20)$$

This observation will be relevant in Eq. (25) below. Second, the theta series (17) assumes a choice of Lagrangian decomposition $\Gamma = \Gamma_e \oplus \Gamma_m$. Under a monodromy in $\mathcal{M}_c(\mathcal{X})$, this choice will generally not be preserved. The partition function $\mathcal{Z}_\Theta(C)$ will nevertheless be invariant (after the necessary transformation of C , Θ and the period matrix \mathcal{N}) provided $\Psi(\zeta)$ stays invariant under the action of $\rho(M)$ via the metaplectic (Schrödinger-Weil) representation. This indeed holds for the Gaussian solution (18), and provides a strong constraint on its non-Gaussian generalization. It also suggests that there should be a direct relation between the exact $\Psi(\zeta)$ and the topological string wave-function, which we spell out in [1].

TOPOLOGY OF THE NS-AXION CIRCLE BUNDLE \mathcal{C} OVER \mathcal{T}

Having recalled some basic properties of the fivebrane partition function \mathcal{Z}_Θ , we can now analyze the implications of the instanton correction (1) for the topological nature of the NS-axion σ . The first, obvious observation is that (1) breaks continuous shift symmetries $\sigma \mapsto \sigma + 2\kappa$ to those with integer κ . Thus, $e^{i\pi\sigma}$, $0 \leq \sigma < 2$ parametrizes the fiber of a certain circle bundle \mathcal{C} over the intermediate Jacobian $\mathcal{J}_c(\mathcal{X})$, to be identified.

The second observation is that $e^{i\pi\sigma}$ must transform in the same way as \mathcal{Z}_Θ , Eq. (16), under large gauge transformations. This requires that under $C \mapsto C + H$, σ should simultaneously shift according to

$$\sigma \mapsto \sigma - \tilde{\eta}_\Lambda \zeta^\Lambda + \eta^\Lambda \tilde{\zeta}_\Lambda + 2c(H), \quad (21)$$

where $c(H)$ is defined modulo 1 by $\sigma_{\text{NS5}}(H) = (-1)^{2c(H)}$, i.e.

$$c(H) = -\frac{1}{2} \eta^\Lambda \tilde{\eta}_\Lambda + \tilde{\eta}_\Lambda \theta^\Lambda - \eta^\Lambda \phi_\Lambda \pmod{1}. \quad (22)$$

Due to the periodicity of σ modulo 2, the ambiguity of $c(H)$ modulo the addition of integers is irrelevant. Thus, we conclude that large gauge transformations are generated by $T'_{H,\kappa} \equiv T_{H,\kappa+c(H)}$ with $H \in H^3(\mathcal{X}, \mathbb{Z})$ and $\kappa \in \mathbb{Z}$, acting as

$$\begin{aligned} T'_{H,\kappa} : (\zeta^\Lambda, \tilde{\zeta}_\Lambda, \sigma) &\mapsto (\zeta^\Lambda + \eta^\Lambda, \tilde{\zeta}_\Lambda + \tilde{\eta}_\Lambda, \\ \sigma &\mapsto \sigma + 2\kappa - \tilde{\eta}_\Lambda \zeta^\Lambda + \eta^\Lambda \tilde{\zeta}_\Lambda + 2c(H)). \end{aligned} \quad (23)$$

It should be stressed that the extra shift of $c(H)$ is crucial for the consistency of this action, since one finds that $T'_{H_2,\kappa_2} T'_{H_1,\kappa_1} = T'_{H_1+H_2,\kappa_3}$ where

$$\kappa_3 = \kappa_1 + \kappa_2 + c(H_1) + c(H_2) + \frac{1}{2} \langle H_1, H_2 \rangle - c(H_1 + H_2) \quad (24)$$

is an integer, by virtue of (14). The extra shift in σ was recently observed in the context of rigid CY compactifications upon assuming invariance under a certain natural arithmetic group [27], and, with hindsight, could also have been detected in a similar construction in the non-rigid case [28].

At this point, we can now explain why the equality of the two quadratic refinements σ_D and σ_{NS5} is desirable. On the one hand, using (20), one sees that a change $\Theta \mapsto \Theta'$ of the characteristics governing fivebrane instanton corrections leaves (1) invariant provided one redefines the axions into

$$\hat{C} = C + \Theta - \Theta', \quad \hat{\sigma} = \sigma + \langle \Theta - \Theta', C \rangle - \langle \Theta, \Theta' \rangle. \quad (25)$$

On the other hand, a change $\Theta_D \mapsto \Theta'_D$ of the characteristics Θ_D governing the D-instanton contributions leaves (10) invariant provided it is accompanied by a similar field redefinition (25) where Θ, Θ' are replaced by Θ_D, Θ'_D . These two field redefinitions are compatible as long as $\Theta - \Theta_D = \Theta' - \Theta'_D$. Thus, if one wants to ensure that physics (in particular the moduli space \mathcal{M}) is independent of the choice of quadratic refinement, as is known to be the case in $\mathcal{N} = 2$ gauge theories [6, 29], one must require that the difference $\Theta - \Theta_D$ is fixed. Since the difference of two characteristics transforms (modulo integers) like a symplectic vector under monodromies, and since the monodromy group in general does not admit any invariant symplectic vector, the most natural choice is to set $\Theta = \Theta_D$. This equality is also generally expected from S-duality, which relates NS5- and D5-brane instantons on the type IIB side.

TOPOLOGY OF THE NS-AXION CIRCLE BUNDLE \mathcal{C} OVER $\mathcal{M}_c(\mathcal{X})$

So far we have established the fibration of the NS-axion circle bundle \mathcal{C} over the torus of C -fields. The next question is to understand the fibration of this bundle over the complex structure moduli space $\mathcal{M}_c(\mathcal{X})$. This is obviously tied with the metric-dependent normalization factor \mathcal{F} in the fivebrane partition function, which is notoriously subtle [30]. We shall limit ourselves to some preliminary comments in this direction.

To this aim, let us return to the perturbative metric (2), and compute the curvature of the connection (5) on the circle bundle \mathcal{C} . Taking into account that σ has periodicity two, and the value (6) of the one-loop parameter c , we then find

$$d\left(\frac{D\sigma}{2}\right) = \omega_{\mathcal{T}} + \frac{\chi(\mathcal{X})}{24} \omega_{S\mathcal{K}}, \quad (26)$$

where $\omega_{\mathcal{T}}$ is, as before, the Kähler form on \mathcal{T} , and $\omega_{S\mathcal{K}} = -\frac{1}{2\pi} d\mathcal{A}_K$ is the Kähler form on the complex structure moduli space $\mathcal{M}_c(\mathcal{X})$. The first term in (26) indeed confirms our identification of $\mathcal{C}|_{\mathcal{T}}$ with the fivebrane circle bundle \mathcal{C}_Θ .

The second term in (26) suggests that $e^{i\pi\sigma}$ is a section of $\mathcal{L}^{\frac{\chi(\mathcal{X})}{24}}$, where \mathcal{L} is the Hodge line bundle over $\mathcal{M}_c(\mathcal{X})$, i.e. the bundle whose sections transform as $s \mapsto s e^f$ under rescalings $\Omega_{3,0} \mapsto e^f \Omega_{3,0}$ of the holomorphic 3-form on \mathcal{X} . This however causes trouble, since $\chi(\mathcal{X})$ is rarely a multiple of 24, and \mathcal{L} does not admit any natural 24:th root. In particular, f_M in Eq. (9) is only defined modulo $2\pi i$, which implies that the shift of σ under monodromies is ambiguous modulo $\chi(\mathcal{X})/12$. Since fivebrane charge quantization requires that σ is periodic modulo 2, there must be an additional constant shift of σ in Eq. (9) to resolve this ambiguity. A similar problem afflicts the topological string amplitude of the B-model on \mathcal{X} , which is claimed to transform as a section of $\mathcal{L}^{\frac{\chi(\mathcal{X})}{24}-1}$ under monodromies [31]. A proper understanding of the topological nature of either the axion σ , the topological string amplitude or the fivebrane partition function presumably involves determinant line bundles, along the lines of [11, 32], and should allow to compute the variation of the NS-axion under monodromies. We hope to address this issue in future work.

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